# On Coresets for Regularized Loss Minimization

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## Abstract

We design and mathematically analyze sampling-based algorithms for regularized loss minimization problems that are implementable in popular computational models for large data, in which the access to the data is restricted in some way. Our main result is that if the regularizer's effect does not become negligible as the norm of the hypothesis scales, and as the data scales, then a uniform sample of modest size is with high probability a coreset. In the case that the loss function is either logistic regression or soft-margin support vector machines, and the regularizer is one of the common recommended choices, this result implies that a uniform sample of size $O(d\sqrt{n})$ is with high probability a coreset of $n$ points in $\Re^d$. We contrast this upper bound with two lower bounds. The first lower bound shows that our analysis of uniform sampling is tight; that is, a smaller uniform sample will likely not be a core set. The second lower bound shows that in some sense uniform sampling is close to optimal, as significantly smaller core sets do not generally exist.

## 1 Introduction

We consider the design and mathematical analysis of sampling-based algorithms for regularized loss minimization (RLM) problems on large data sets [23]. The input consists of a collection $X = \{x_1, x_2, \ldots, x_n\}$ of points in $\Re^d$, and a collection $Y = \{y_1, y_2, \ldots, y_n\}$ of associated labels from $\{-1, 1\}$. Intuitively the goal is to find a hypothesis $\beta \in \Re^d$ that is the best "linear" explanation for the labels. More formally, the objective is to minimize a function $F(\beta)$ that is a linear combination of a nonnegative nondecreasing loss function $\ell$ that measures the goodness of the hypothesis, and a nonnegative regularization function $r$ that measures the complexity of the hypothesis. So:

$$F(\beta) = \sum_{i=1}^{n} \ell(-y_i \beta \cdot x_i) + \lambda \, r(R\beta) \tag{1}$$

Notable examples include regularized logistic regression, where the loss function is $\ell(z) = \log(1 + \exp(z))$, and regularized soft margin support vector machines (SVM), where the loss function is $\ell(z) = \max(0, 1 + z)$.

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Common regularizers are the 1-norm, the 2-norm, and the 2-norm squared [3]. The parameter $\lambda \in \Re$ is ideally set to balance the risks of over-fitting and under-fitting. It is commonly recommended to set $\lambda$ to be proportional to $\sqrt{n}$ [20, 23]. For this choice of $\lambda$, if there was a true underlying distribution from which the data was drawn in an i.i.d. manner, then there is a guarantee that the computed $\beta$ will likely have vanishing relative error with respect to the ground truth [23, Corollary 13.9] [20, Corollary 3]. We will generalize this somewhat and assume that $\lambda$ is proportional to $n^\kappa$ for some $0 < \kappa < 1$. The parameter $R$ is the maximum 2-norm of any point in $X$. Note that the regularizer must scale with $R$ if it is to avoid having a vanishing effect as the point set $X$ scales.[1]

We are particularly interested in settings where the data set is too large to fit within the main memory of one computer, and thus the algorithm's access to the data set is restricted in some way. Popular computation models that arise from such settings include:

**Streaming Model:** This model derives from settings where the data is generated in real-time, or stored on a memory technology (such as a disk or tape) where a sequential scan is way more efficient than random accesses. In this model the data can only be accessed by a single (or a small number of) sequential passes [19].

**Massively Parallel Computation (MPC) Model:** This model derives from settings where the data is distributed over multiple computers. In this model only a few rounds of communication with sublinear sized messages are allowed [10, 11].

**Relational Model:** This model derives from settings where the data is stored in a database in a collection of tables. In this model the data must be accessed via relational operators that do not explicitly join tables [12].

Thus we additionally seek algorithms that can be reasonably implemented in these popular restricted access models.

One popular method to deal with large data sets is to extract a manageably small (potentially weighted) sample from the data set, and then directly solve (a weighted version of) the RLM problem on the (weighted) sample[2]. The aspiration here is that the optimal solution on the sample will be a good approximation to the optimal solution on the original data set. To achieve this aspiration, the probability that a particular point is sampled (and the weight that it is given) may need to be carefully computed as some points may be more important than other points. But if this sampling probability distribution is too complicated, it may not be efficiently implementable in common restricted access models.

A particularly strong condition on the sample that is sufficient for achieving this aspiration is that the sample is a *coreset*; intuitively, a sample is a coreset if *for all possible hypotheses* $\beta$, the objective value of $\beta$ on the sample is very close to the objective value of $\beta$ on the whole data set.

Our original research goal was to determine when small coresets exist for RLM problems in general, and for regularized logistic regression and regularized SVM in particular, and when these coresets can be efficiently computed within the common restricted access models.

**Our Results:** Our main result, covered in Section 3, is that if the regularizer's effect does not become negligible as the norm of the hypothesis scales then a uniform sample of size $\Theta(n^{1-\kappa}\Delta)$ points is with high probability a coreset. Here, $\Delta$ is the VC-dimension of the loss function. Formally this scaling condition says that if $\ell(-\|\beta\|) = 0$ then $r(\beta)$ must be a constant fraction of $\ell(\|\beta\|_2)$. We show that this scaling condition holds when the loss function is either logistic regression or SVM, and the regularizer is the 1-norm, the 2-norm, or the 2-norm squared. So for example, in the standard case that $\kappa = 1/2$, the scaling condition ensures that a uniform sample of $\tilde{\Theta}(d\sqrt{n})$ points is with high probability a coreset when the regularizer is one of the standard ones, and the loss function is either logistic regression and SVM, as they have VC-dimension $O(d)$. Note also that uniform sampling can be reasonably implemented in all of the popular restricted access

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[1]To see this note that if we multiplied each coordinate of each point $x_i$ by a factor of $c$, the optimal hypothesis $\beta$ would decrease by a factor of $c$, thus decreasing the value of all of the standard regularizers.

[2]A more general approach is to summarize the data set in some more sophisticated way than as a weighted sample, but such approaches are beyond the scope of this paper.

models. So this yields a reasonable algorithm for all of the restricted access models under the assumption that a data set of size $\tilde{\Theta}(d\sqrt{n})$ can be stored, and reasonably solved, in the main memory of one computer.

We complement our upper bound with two lower bounds on the size of coresets. Our lower bounds assume the 2-norm squared as the regularizer, since intuitively this is the standard regularizer for which it should be easiest to attain small coresets. We first show in Section 4 that our analysis is asymptotically tight for uniform sampling. That is, we show that for both logistic regression and SVM, a uniform sample of size $O(n^{1-\kappa-\epsilon})$ may not result in a coreset. We then show in Section 5 that for both logistic regression and SVM there are instances in which every core set is of size $\Omega(n^{(1-\kappa)/5-\epsilon})$. So more sophisticated sampling methods must still have core sets whose size is in the same ballpark as is needed for uniform sampling. One might arguably summarize our results as saying that the simplest possible sampling method is nearly optimal for obtaining a coreset.

Finally in Section 6 we experimentally evaluate the practical utility of uniform sampling for logistic regression using several real-world datasets from the UCI machine learning dataset repository [6]. We observe that our theory is empirically validated as uniform samples yield good empirical approximation, and orders-of-magnitude speedup over learning on the full dataset.

**Related Work on Coresets:** The most closely related prior work is probably [18], who considered coresets for *unregularized* logistic regression; i.e, the regularization parameter $\lambda = 0$. [18] showed that are data sets for which there do not exist coresets of sublinear size, and then introduced a parameter $\mu$ of the instances that intuitively is small when there is no hypothesis that is a good explanation of the labels, and showed that a coreset of size roughly linear in $\mu$ can be obtain by sampling each point with a uniform probability plus a probability proportional to its $\ell_2^2$ leverage scores (which can be computed from a singular value decomposition of the points). This result yields an algorithm, for the promise problem in which $\mu$ is known a priori to be small (but it is not clear how to reasonably compute $\mu$), that is reasonably implementable in the MPC model, and with two passes over the data in the streaming model. It seems unlikely that this algorithm is implementable in the relational model due to the complex nature of required sampling probabilities. Contemporaneously with our research, [24] obtained results similar in flavor to those of [18]. [24] also show that small coresets exist for certain types of RLM instances; in this case, those in which the norm of the optimal hypothesis is small. So for normalized logistic regression [24] shows that when the 2-norm of the optimal $\beta$ is bound by $\mu$, coresets of size $\tilde{O}(\mu^2 n^{1-\kappa})$ can be obtained by sampling a point with probability proportional to its norm divided by its ordinal position in the sorted order of norms. So again this yields an algorithm for the promise problem in which $\mu$ is known a priori to be small (and again it is not clear how to reasonably compute $\mu$). Due to the complex nature of the probabilities it is not clear that this algorithm is reasonably implementable in any of the restricted access models that we consider. So from our perspective there are three key differences between the results of [18] and [24] and our positive result: *(1)* our result applies to all data sets *(2)* we use uniform sampling, and thus *(3)* our sampling algorithm is implementable in all of the restricted access models that we consider.

Surveys of the use of coresets in algorithmic design can be found in [17] and in [9, Chapter 23]. The knowledge that sampling with probability at least proportional to sensitivity yields a coreset has been used for at least a decade as it is used by [5]. Coresets were used for partitioned clustering problems, such as $k$-means [1, 8, 15]. Coresets were also used the Minimum Enclosing Ball (MEB) problem [9]. Coresets for MEB are the basis for the Core Vector Machine approach to unregularized kernelized SVM [25]. We note that while there is a reduction from kernelized SVM to MEB, the reduction is not approximation preserving, and thus the existence of coresets for MEB does not necessarily imply the existence of coresets for SVM. Coresets have also been used for submodular optimization [16], and in the design of streaming algorithms (e.g. [21]), as well as distributed algorithms (e.g. [14]).

## 2 Preliminaries

We define $\ell_i(\beta) = \ell(-y_i\beta \cdot x_i)$ as the contribution of point $i$ to the loss function. We define $f_i(\beta) = \ell(-y_i\beta \cdot x_i) + \lambda r(R\beta)/n$ as the contribution of point $i$ to the objective $F(\beta)$. The sensitivity of point $i$ is then $s_i = \sup_\beta f_i(\beta)/F(\beta)$, and the total sensitivity is $S = \sum_{i=1}^n s_i$. For $\epsilon > 0$, an $\epsilon$-coreset $(C, U)$ consists

of a subcollection $C$ of $[1, n]$, and associated nonnegative weights $U = \{u_i \mid i \in C\}$, such that

$$\forall \beta \quad H(\beta) := \frac{\left| \sum_{i=1}^n f_i(\beta) - \sum_{i \in C} u_i f_i(\beta) \right|}{\sum_{i=1}^n f_i(\beta)} \leq \epsilon \tag{2}$$

Conceptually one should think of $u_i$ as a multiplicity, that is that $x_i$ is representing $u_i$ points from the original data set. So one would expect that $\sum_{i \in C} u_i = n$, although this is not strictly required. But it is easy to observe that $\sum_{i \in C} u_i$ must be close to $n$.

**Observation 1.** *Assume that $\ell(0) \neq 0$, as is the case for logistic regression and SVM. If $(C, U)$ is an $\epsilon$-coreset then $(1 - \epsilon)n \leq \sum_{i \in C} u_i \leq (1 + \epsilon)n$.*

*Proof.* Applying the definition of coreset in the case that $\beta$ is the hypothesis with all 0 components, it must be the case that $\left| \sum_{i=1}^n \ell(0) - \sum_{i \in C} u_i \ell(0) \right| \leq \epsilon \sum_{i=1}^n \ell(0)$, or equivalently $\left| n - \sum_{i \in C} u_i \right| \leq \epsilon n$. $\qquad\square$

Note that in the special case that each $u_i$ is equal to a common value $u$, as will be the case for uniform sampling, setting each $u_i = 1$ and scaling $\lambda$ down by a factor of $u$, would result in the same optimal hypothesis $\beta$.

A collection $X$ of data points is *shatterable* by a loss function $\ell$ if for every possible set of assignments of labels, there is a hypothesis $\beta$ and a threshold $t$, such that for the positively labeled points $x_i \in X$ it is the case the $\ell(\beta \cdot x_i) \geq t$, and for the negatively labeled points $x_i$ it is the case that $\ell(\beta \cdot x_i) < t$. The VC-dimension of a loss function is then the maximum cardinality of a shatterable set. It is well known that if the loci of points $x \in \Re^d$ where $\ell(\beta \cdot x) = t$ is a hyperplane then the VC-dimension is at most $d + 1$ [26]. It is obvious that this property holds if the loss function is SVM, and [17] show that it holds if the loss function is logistic regression. The regularizer does not affect the VC-dimension of a RLM problem.

A loss function $\ell$ and a regularizer $r$ satisfy the $(\sigma, \tau)$-scaling condition if $\ell(-\sigma) > 0$, and if $\|\beta\|_2 \geq \sigma$ then $r(\beta) \geq \tau \, \ell(\|\beta\|_2)$.

**Theorem 2** ( [2, 7]). *Let $(n, X, Y, \ell, r, \lambda, R, \kappa)$ be an instance of the RLM problem where the loss function has VC-dimension at most $\Delta$. Let $s_i'$ be an upper bound on the sensitivity $s_i$, let $S' = \sum_P s_i'$. Let $\epsilon, \delta \in (0, 1)$ be arbitrary. Let $C \subseteq [1, n]$ be a random sample of at least $\frac{10S'}{\epsilon^2}(\Delta \log S' + \log(\frac{1}{\delta})))$ points sampled in an i.i.d fashion, where the probability that point $i \in [1, n]$ is selected each time is $s_i'/S'$. Let the associated weight $u_i$ for each point $i \in C$ be $\frac{S'}{s_i' |C|}$. Then $C$ and $U = \{u_i \mid i \in C\}$ is an $\epsilon$-coreset with probability at least $(1 - \delta)$,.*

# 3    Upper Bound for Uniform Sampling

**Theorem 3.** *Let $(n, X, Y, \ell, r, \lambda, R, \kappa)$ be an instance of the RLM problem where $\ell$ and $r$ satisfy the $(\sigma, \tau)$-scaling condition and the loss function has VC-dimension at most $\Delta$. Let $S' = \frac{n}{\tau \lambda} + \frac{\ell(\sigma)}{\ell(-\sigma)} + 1$. A uniform sample of $q = \frac{10S'}{\epsilon^2}(\Delta \log S' + \log(\frac{1}{\delta}))$ points, each with an associated weight of $u = n/q$, is an $\epsilon$-coreset with probability at least $1 - \delta$.*

*Proof.* With an aim towards applying Theorem 2 we start by upper bounding the sensitivity of an arbitrary point. To this end consider an arbitrary $i \in [1, n]$ and an arbitrary hypothesis $\beta$. First consider the case

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that $R\|\beta\|_2 \geq \sigma$. In this case:

$$\frac{f_i(\beta)}{F(\beta)} = \frac{\ell(-y_i\beta \cdot x_i) + \frac{\lambda}{n}r(R\beta)}{\sum_j \ell(-y_j\beta \cdot x_j)) + \lambda\, r(R\beta)}$$

$$\leq \frac{\ell(|\beta \cdot x_i|) + \frac{\lambda}{n}r(R\beta)}{\sum_j \ell(-y_j\beta \cdot x_j) + \lambda\, r(R\beta)} \qquad \text{As the loss function is nondecreasing}$$

$$\leq \frac{\ell(|\beta \cdot x_i|) + \frac{\lambda}{n}r(R\beta)}{\lambda\, r(R\beta)} \qquad \text{As the loss function is nonnegative}$$

$$\leq \frac{\ell(|\beta \cdot \beta| \frac{R}{\|\beta\|_2}) + \frac{\lambda}{n}r(R\beta)}{\lambda\, r(R\beta)} \qquad \text{As maximum is when } x_i = \beta\frac{R}{\|\beta\|_2}$$

$$\leq \frac{\ell(R\,\|\beta\|_2)}{\lambda\, r(R\beta)} + \frac{1}{n}$$

$$\leq \frac{\ell(R\,\|\beta\|_2)}{\lambda\, \tau\, \ell(R\,\|\beta\|_2)} + \frac{1}{n} \qquad \text{By } (\sigma,\tau) \text{ scaling assumption and assumption } R\|\beta\|_2 \geq \sigma$$

$$\leq \frac{1}{\tau\lambda} + \frac{1}{n}$$

Next consider the case that $R\|\beta\|_2 < \sigma$. In this case:

$$\frac{f_i(\beta)}{F(\beta)} = \frac{\ell(-y_i\beta \cdot x_i) + \frac{\lambda}{n}r(R\beta)}{\sum_j \ell(-y_j\beta \cdot x_j) + \lambda\, r(R\beta)}$$

$$\leq \frac{\ell(|\beta \cdot x_i|) + \frac{\lambda}{n}r(R\beta)}{\sum_j \ell(-|\beta \cdot x_j|) + \lambda r(R\beta)} \qquad \text{As the loss function is nondecreasing}$$

$$\leq \frac{\ell(|\beta \cdot \beta| \frac{R}{\|\beta\|_2}) + \frac{\lambda}{n}r(R\beta)}{\sum_j \ell(-|\beta \cdot \beta| \frac{R}{\|\beta\|_2}) + \lambda\, r(R\beta)} \qquad \text{As maximum is when } x_i = \beta\frac{R}{\|\beta\|_2}$$

$$\leq \frac{\ell(R\,\|\beta\|_2) + \frac{\lambda}{n}r(R\beta)}{\sum_j \ell(-R\,\|\beta\|_2) + \lambda\, r(R\beta)}$$

$$\leq \frac{\ell(R\,\|\beta\|_2)}{\sum_j \ell(-R\,\|\beta\|_2)} + \frac{1}{n} \qquad \text{As } a,b,c,d \geq 0 \text{ implies } \frac{a+b}{c+d} \leq \frac{a}{c} + \frac{b}{d}$$

$$\leq \frac{\ell(\sigma)}{\sum_j \ell(-\sigma)} + \frac{1}{n} \qquad \text{By assumption } R\|\beta\|_2 < \sigma$$

$$\leq \frac{\ell(\sigma)}{n\, \ell(-\sigma)} + \frac{1}{n}$$

Thus the sensitivity of every point is at most $\frac{1}{\tau\lambda} + \frac{\ell(\sigma)}{n\,\ell(-\sigma)} + \frac{1}{n}$, and the total sensitivity $S$ is at most $\frac{n}{\tau\lambda} + \frac{\ell(\sigma)}{\ell(-\sigma)} + 1$. The claim the follows by Theorem 2. $\qquad\square$

**Corollary 4.** *Let $(n, X, Y, \ell, r, \lambda, R, \kappa)$ be an instance of the RLM problem where the loss function $\ell$ is logistic regression or SVM, and the regularizer $r$ is one of the 1-norm, 2-norm, or 2-norm squared. Let $S' = \frac{12n}{\lambda} + 6 = 12n^{1-\kappa} + 6$. A uniform sample of $q = \frac{10S'}{\epsilon^2}((d+1)\log S' + \log(\frac{1}{\delta})))$ points, each with an associate weight of $u = \frac{n}{q}$, is an $\epsilon$-coreset with probability at least $1 - \delta$.*

*Proof.* Since the VC-dimension of logistic regression and SVM is at most $d+1$, it is enough to show that the scaling condition holds in each case. First consider logistic regression. Let $\sigma = 1$. Then we have $l(-1) = \log(1 + \exp(-1)) \neq 0$. In the case that $r(\beta) = \|\beta\|_2$ it is sufficient to take $\tau = \frac{1}{2}$ as $\ell(z) = \log(1 + \exp(z)) \leq 2z$ when $z \geq 1$. Similarly its sufficient to take $\tau = \frac{1}{2}$ when the regularizer is the 2-norm

squared, as $\ell(z) = \log(1 + \exp(z)) \leq 2z^2$ when $z \geq 1$. As $\|\beta\|_1 \geq \|\beta\|_2$ it is also sufficient to take $\tau = \frac{1}{2}$ when the regularizer is the 1-norm. Therefore, total sensitivity is bounded by $\frac{2n}{\lambda} + 6$ in all of these cases.

Now consider SVM. Let $\sigma = 1/2$. Then $l(-1/2) = 1/2 \neq 0$. In the case that $r(\beta) = \|\beta\|_2$ it is sufficient to take $\tau = \frac{1}{3}$ as $\ell(z) = 1 + z \leq 3z$ when $z \geq \frac{1}{2}$; $\tau = \frac{1}{3}$ will be also sufficient when the regularizer is the 1-norm since $\|\beta\|_1 \geq \|\beta\|_2$.

Furthermore, if $\|\beta\|_2 \geq 1$, then $\|\beta\|_2^2 \geq 4\|\beta\|_2$; therefore, in the case that $r(\beta) = \|\beta\|_2^2$, it is sufficient to take $\tau = \frac{1}{12}$. Therefore, total sensitivity is bounded by $\frac{12n}{\lambda} + 4$. $\qquad\square$

The implementation of uniform sampling, and the computation of $R$, in the streaming and MPC models is trivial. Uniform sampling and the computation of $R$ in the relational model can be implemented without joins because both can be expressed using *functional aggregate queries*, which can then be efficiently computed without joins [12].

# 4 Uniform Sampling Lower Bound

In this section we show in Theorem 5 that our analysis of uniform sampling is tight up to poly-logarithmic factors.

**Theorem 5.** *Assume that the loss function is either logistic regression or SVM, and the regularizer is the 2-norm squared. Let $\epsilon, \gamma \in (0, 1)$ be arbitrary. For all sufficiently large $n$, there exists an instance $I_n$ of $n$ points such that with probability at least $1 - 1/n^{\gamma/2}$ it will be the case that for a uniform sample $C$ of $c = n^{1-\gamma}/\lambda = n^{1-\kappa-\gamma}$ points, there is no weighting $U$ that will result in an $\epsilon$-coreset.*

*Proof.* The instance $I_n$ consists of points located on the real line, so the dimension $d = 1$. A collection $A$ of $n - (\lambda n^{\gamma/2})$ points is located at $+1$, and the remaining $\lambda n^{\gamma/2}$ points are located at $-1$; call this collection of points $B$. All points are labeled $+1$. Note $R = 1$.

Let $C$ be the random sample of $c$ points, and $U$ an arbitrary weighting of the points in $C$. Note that $U$ may depend on the instantiation of $C$. Our goal is to show that with high probability, $(C, U)$ is not an $\epsilon$-coreset. Our proof strategy is to first show that because almost all of the points are in $A$, it is likely that $C$ contains only points from $A$. Then we want to show that, conditioned on $C \subseteq A$, that $C$ can not be a coreset for any possible weighting. We accomplish this by showing that $\lim_{n\to\infty} H(\beta) = 1$ when $\beta = n^{\gamma/4}$.

We now show that one can use a standard union bound to establish that it is likely that $C \subseteq A$. To accomplish this let $E_i$ be the probability that the the $i^{th}$ point selected to be in $C$ is not in $A$.

$$\Pr[C \subseteq A] = 1 - \Pr\left[\vee_{i \in C} E_i\right] \geq 1 - |C|\frac{|B|}{n} = 1 - \frac{n^{1-\gamma}}{\lambda}\frac{\lambda n^{\gamma/2}}{n} = 1 - \frac{1}{n^{\gamma/2}}$$

Now we show if $C \subseteq A$ and $n$ is large enough, then $(C, U)$ cannot be an $\epsilon$-coreset for any collection $U$ of weights. To accomplish this consider the the hypothesis $\beta_0 = n^{\gamma/4}$. From the definition of coreset, it is sufficient to show that $H(\beta_0)$, defined as,

$$H(\beta_0) = \frac{|\sum_{i \in P} f_i(\beta_0) - \sum_{i \in C} u_i f_i(\beta_0)|}{\sum_{i \in P} f_i(\beta_0)} \tag{3}$$

is greater than $\epsilon$. We accomplish this by showing that the limit as n goes to infinity of $H(\beta_0)$ is 1. Applying Condition 1 we can conclude that

$$H(\beta_0) \geq \frac{|\sum_{i \in P} \ell_i(\beta_0) - \sum_{i \in C} u_i \ell_i(\beta_0)| - \epsilon\lambda\|\beta_0\|_2^2}{\sum_{i \in P} \ell_i(\beta_0) + \lambda\|\beta_0\|_2^2} \tag{4}$$

Then, using the fact that $A$ and $B$ is a partition of the points and $C \subseteq A$ we can conclude that

$$H(\beta_0) \geq \frac{|\sum_{i \in A} \ell_i(\beta_0) + \sum_{i \in B} \ell_i(\beta_0) - \sum_{i \in C} u_i \ell_i(\beta_0)| - \epsilon \lambda \|\beta_0\|_2^2}{\sum_{i \in A} \ell_i(\beta_0) + \sum_{i \in B} \ell_i(\beta_0) + \lambda \|\beta_0\|_2^2}$$

$$= \frac{\left| \frac{\sum_{i \in A} \ell_i(\beta_0)}{\sum_{i \in B} \ell_i(\beta_0)} + 1 - \frac{\sum_{i \in C} u_i \ell_i(\beta_0)}{\sum_{i \in B} \ell_i(\beta_0)} \right| - \frac{\epsilon \lambda \|\beta_0\|_2^2}{\sum_{i \in B} \ell_i(\beta_0)}}{\frac{\sum_{i \in A} \ell_i(\beta_0)}{\sum_{i \in B} \ell_i(\beta_0)} + 1 + \frac{\lambda \|\beta_0\|_2^2}{\sum_{i \in B} \ell_i(\beta_0)}} \tag{5}$$

We now need to bound various terms in equation (5). Let us first consider logistic regression. Note that

$$\sum_{i \in B} \ell_i(\beta_0) = |B| \log(1 + \exp(n^{\gamma/4})) \geq |B| n^{\gamma/4} = \lambda n^{3\gamma/4} \tag{6}$$

Therefore,

$$\lim_{n \to \infty} \frac{\lambda \|\beta_0\|_2^2}{\sum_{i \in B} \ell_i(\beta_0)} \leq \lim_{n \to \infty} \frac{\lambda n^{\gamma/2}}{\lambda n^{3\gamma/4}} = 0 \tag{7}$$

Also note that

$$\lim_{n \to \infty} \sum_{i \in A} \ell_i(\beta_0) = \lim_{n \to \infty} |A| \log(1 + \exp(-n^{\gamma/4})) \leq \lim_{n \to \infty} n \exp(-n^{\gamma/4}) = 0 \tag{8}$$

Finally, by Observation 1, we have,

$$\lim_{n \to \infty} \sum_{i \in C} u_i \ell_i(\beta_0) \leq \lim_{n \to \infty} (1 + \epsilon) n \exp(-n^{\gamma/4}) = 0 \tag{9}$$

Combining equations (6), (7), (8), and (9), we the expression in equation (5) converges to 1 as $n \to \infty$. Thus for sufficiently large $n$, $H(\beta_0) > \epsilon$ and thus $(C, U)$ is not an $\epsilon$-coreset.

We now need to bound various terms in equation (5) for SVM. First note that

$$\sum_{i \in B} \ell_i(\beta_0) = |B|(1 + n^{\gamma/4}) \geq |B| n^{\gamma/4} = \lambda n^{3\gamma/4} \tag{10}$$

Therefore,

$$\lim_{n \to \infty} \frac{\lambda \|\beta_0\|_2^2}{\sum_{i \in B} \ell_i(\beta_0)} \leq \lim_{n \to \infty} \frac{\lambda n^{\gamma/2}}{\lambda n^{3\gamma/4}} = 0 \tag{11}$$

Also note that

$$\lim_{n \to \infty} \sum_{i \in A} \ell_i(\beta_0) = \lim_{n \to \infty} |A| \max(0, 1 - n^{\gamma/4}) = 0 \tag{12}$$

Finally, by Observation 1, we have that:

$$\lim_{n \to \infty} \sum_{i \in C} u_i \ell_i(\beta_0) \leq \lim_{n \to \infty} (1 + \epsilon) n \max(0, 1 - n^{\gamma/4}) = 0 \tag{13}$$

Combining equations (10), (11), (12), and (13), we the expression in equation (5) converges to 1 as $n \to \infty$. Thus for sufficiently large $n$, $H(\beta_0) > \epsilon$ and thus $(C, U)$ is not an $\epsilon$-coreset. $\quad\square$

# 5 General Lower Bound on Coreset Size

This section is devoted to proving the following theorem:

**Theorem 6.** *Assume that the loss function is either logistic regression or SVM, and the regularizer is the 2-norm squared. Let $\epsilon, \gamma \in (0,1)$ be arbitrary. For all sufficiently large $n$, there exists an instance $I_n$ of $n$ points such that $I_n$ does not have an $\epsilon$-coreset of size $O(n^{(1-\kappa)/5-\gamma})$*

## 5.1 Logistic Regression

The goal in this subsection is to prove Theorem 6 when the loss function is logistic regression. The lower bound instance $I_n$ consists of a collection of $n$ positively-labeled points in $\Re^3$ uniformly spaced around a circle of radius 1 centered at $(0,0,1)$ in the plane $z = 1$. Note that $R = \sqrt{2}$. However for convenience, we will project $I_n$ down into a collection $X$ of points in the plane $z = 0$. So the resulting instance, which we call the circle instance, consists of $n$ points uniformly spread around the unit circle in $\Re^2$. So for a hypothesis $\beta = (\beta_x, \beta_y, \beta_z)$, $F(\beta)$ is now $\sum_{x_i \in X} \ell(-y_i((\beta_x, \beta_y) \cdot x_i + \beta_z)) + 2\lambda \|\beta\|_2^2$. So $\beta_z$ can be thought of as an offset or bias term, that allows hypotheses in $\Re^2$ that do not pass through the origin.

Fix a constant $c > 0$ and a subset $C$ of $X$ that has size $k = c\frac{n^{1/5-\gamma}}{\lambda^{1/5}} = cn^{(1-\kappa)/5-\gamma}$ as a candidate coreset. Let $U$ be an arbitrary collection of associated weights. Toward finding a hypothesis that violates equation (2), define a *chunk* $A$ to be a collection of $\frac{n}{4k}$ points in the middle of $\frac{n}{2k}$ consecutive points on the circle that are all not in $C$. So no point in the chunk $A$ is in $C$, and no point in the next $\frac{n}{8k}$ points in either direction around the circle are in $C$. Its easy to observe that, by the pigeon principle, a chunk $A$ must exist. Now let $\beta_A = (\beta_x, \beta_y, \beta_z)$ be the hypothesis where $(\beta_x, \beta_y) \cdot x_i + \beta_z = 0$ for the two points $x_i \in X \setminus A$ that are adjacent to the chunk $A$, that predicts $A$ incorrectly (and thus that predicts the points $X \setminus A$ correctly), and where $\|\beta_A\|_2 = \sqrt{\frac{n^{1-\gamma}}{k\lambda}}$. To establish Theorem 6 we want to show that equation (2) is not satisfied for the hypothesis $\beta_A$. By Observation 1 it is sufficient to show that the limit as $n \to \infty$ of:

$$\frac{\left| \sum_{x_i \in X} \ell_i(\beta_A) - \sum_{x_i \in C} u_i \ell_i(\beta_A) \right| - 2\epsilon\lambda \|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A) + \lambda \|\beta_A\|_2^2} = \frac{\left| 1 - \frac{\sum_{x_i \in C} u_i \ell_i(\beta_A)}{\sum_{x_i \in X} \ell_i(\beta_A)} \right| - \frac{2\epsilon\lambda \|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A)}}{1 + \frac{\lambda \|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A)}}$$

is 1. To accomplish this it is sufficient to show that the limits of the ratios in the second expression approach 0, which we do in the next two lemmas.

**Lemma 7.** $\lim_{n \to \infty} \frac{\lambda \|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A)} = 0.$

*Proof.* As the $\frac{n}{4k}$ points in $A$ have been incorrectly classified by $\beta_A$, we know that $\ell_i(\beta_A) \geq \log 2$ for $x_i \in A$. Thus:

$$\lim_{n \to \infty} \frac{\lambda \|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A)} \leq \lim_{n \to \infty} \frac{\lambda \frac{n^{1-\gamma}}{k\lambda}}{\frac{n}{4k} \log 2} = \lim_{n \to \infty} \frac{4}{n^\gamma \log 2} = 0$$

$\square$

Let $d_i$ be the distance between $x_i$ and the line that passes through the first and last points in the chunk $A$. Let $\theta_i$ be the angle formed by the the ray from the origin through $x_i$ and the ray from the origin to them middle point in $A$. Let $\theta = \max_{i \in A} \theta_i = \frac{2\pi}{n} \frac{n}{8k} = \frac{\pi}{4k}$. We then make two algebraic observations.

**Observation 8.** *For all $x_i \in X$, $d_i \|\beta_A\|_2 / 2 \leq |(\beta_x, \beta_y) \cdot x_i + \beta_z| \leq d_i \|\beta_A\|_2$.*

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*Proof.* It is well known that

$$d_i = \frac{|(\beta_x, \beta_y) \cdot x_i + \beta_z|}{\sqrt{\beta_x^2 + \beta_y^2}}$$

Therefore,

$$|(\beta_x, \beta_y) \cdot x_i + \beta_z| = d_i \sqrt{\beta_x^2 + \beta_y^2} \le \|\beta_A\| \, d_i$$

Now we need to show $\|\beta_A\| \, d_i/2 \le |(\beta_x, \beta_y) \cdot x_i + \beta_z|$. Note that there are two points (points adjacent to $A$) $x_j = (a', b')$ for which $(\beta_x, \beta_y) \cdot x_j + \beta_z = 0$. Consider one of them. We have:

$$\begin{aligned}
0 &= \beta_x a' + \beta_y b' + \beta_z \\
&\ge \beta_z - |\beta_x a' + \beta_y b'| \\
&\ge \beta_z - \sqrt{\beta_x^2 + \beta_y^2}\sqrt{a'^2 + b'^2}
\end{aligned}$$

Since the points are over a circle of size 1 we have $\sqrt{a'^2 + b'^2} = 1$. Therefore,

$$\beta_x^2 + \beta_y^2 \ge \beta_z^2$$

So we can conclude:

$$|(\beta_x, \beta_y) \cdot x_i + \beta_z| = d_i \sqrt{\beta_x^2 + \beta_y^2} \ge \frac{d_i}{\sqrt{2}} \|\beta_A\| \ge \frac{d_i}{2} \|\beta_A\|$$

$\square$

**Observation 9.** *For all* $x_i \in X$, $d_i = |\cos(\theta_i) - \cos(\theta)|$.

**Lemma 10.** $\displaystyle \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} = 0.$

*Proof.* We have:

$$\lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)}$$

$$= \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \log\left(1 + \exp(-((\beta_x, \beta_y) \cdot x_i + \beta_z))\right)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)}$$

$$\le \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \log\left(1 + \exp(-\frac{d_i \|\beta_A\|_2}{2})\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)} \qquad \text{By Observation 8}$$

$$\le \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \log\left(1 + \exp(-\frac{\|\beta_A\|_2}{2}(\cos\theta - \cos\theta_i))\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)} \qquad \text{By Observation 9}$$

$$\le \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \exp(-\frac{\|\beta_A\|_2}{2}(\cos\theta - \cos\theta_i))}{\sum\limits_{x_i \in A} \ell_i(\beta_A)} \qquad \text{Since } \log(1 + x) \le x$$

$$\le \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \exp(-\frac{\|\beta_A\|_2}{2}(\cos\frac{\pi}{4k} - \cos\frac{\pi}{2k}))}{\sum\limits_{x_i \in A} \ell_i(\beta_A)} \qquad \text{Since maximizer is when } \theta_i = \frac{\pi}{2k}$$

Using the Taylor expansion of $\cos(x) = \sum_{i=0}^{\infty}(-1)^i \frac{x^{2i}}{(2i)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$, we have $\cos(\frac{\pi}{4k}) - \cos(\frac{\pi}{2k}) \geq$ $\frac{1}{2}\left((\frac{\pi}{2k})^2 - (\frac{\pi}{4k})^2\right) - O(\frac{1}{k^4}) = (\frac{3\pi^2}{32k^2}) - O(\frac{1}{k^4})$. Plugging this inequality, we derive

$$\frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in P} \ell_i(\beta_A)} \leq \frac{\sum\limits_{x_i \in C} u_i \exp(-\frac{\|\beta_A\|}{2}((\frac{3\pi^2}{32k^2}) - O(\frac{1}{k^4})))}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

$$= \frac{\sum\limits_{x_i \in C} u_i \exp(-\frac{n^{2/5}}{2\sqrt{c}\lambda^{2/5}}(\frac{3\pi^2\lambda^{2/5}}{32c^2 n^{2/5-2\gamma}} - O(\frac{\lambda^{4/5}}{n^{4/5-4\gamma}})))}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

$$= \frac{\sum\limits_{x_i \in C} u_i \exp(-\alpha n^{2\gamma} + O(\frac{\lambda^{2/5}}{n^{2/5-4\gamma}}))}{\sum\limits_{x_i \in A} \ell_i(\beta_A)},$$

where $\alpha > 0$ is a constant. Since all the points in $A$ are miss-classified, we have $\ell_i(\beta_A) \geq \log 2$ for all of them. Using this fact and Observation 1, we have:

$$\frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in P} \ell_i(\beta_A)} \leq \frac{(1 + \epsilon)n \exp(-\alpha n^{2\gamma} + O(\frac{\lambda^{2/5}}{n^{2/5-4\gamma}}))}{\frac{n}{4k} \log 2}$$

Finally, using the fact that $k = c\frac{n^{1/5-\gamma}}{\lambda^{1/5}}$ and taking the limit, we conclude:

$$\lim_{n\to\infty} \frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in P} \ell_i(\beta_A)} \leq \lim_{n\to\infty} \frac{4k(1 + \epsilon) \exp(-\alpha n^{2\gamma} + O(\frac{\lambda^{2/5}}{n^{2/5-4\gamma}}))}{\log 2} = 0$$

$\square$

## 5.2  SVM

The goal in this subsection is to prove Theorem 6 when the loss function is SVM. For the sake of contradiction, suppose an $\epsilon$-coreset $(C, u)$ of size $k$ exists for the circle instance. We fix $A$ to be a chunk. Similar to logistic regression, we set $\beta_A$ as the parameters of the linear SVM that separates $A$ from $P/A$ such that the model predicts $A$ incorrectly and predicts the points $P/A$ as positive correctly and $\|\beta_A\|_2 = \sqrt{\frac{n^{1-\gamma}}{k\lambda}} = \frac{n^{2/5}}{\sqrt{c}\lambda^{2/5}}$.

Our goal is to show Eqn. (2) tends to 1 as $n$ grows to infinity. We can break the cost function of Linear SVM into two parts:

$$F_{P,1}(\beta_A) := \sum_{x_i \in P} \ell_i(\beta_A) + 2\lambda \|\beta_A\|_2^2$$

where $\ell_i(\beta_A) = \max(1 - \beta_A x_i y_i, 0) = \max(1 - ((\beta_x, \beta_y) \cdot x_i + \beta_z)y_i, 0)$. Then, we determine the limit of the following quantities as $n$ grows to infinity.

**Lemma 11.** *For the circle instance $P$, if $(C, u)$ is an $\epsilon$-coreset of $P$ with size $k = c\frac{n^{1/5-\gamma}}{\lambda^{1/5}}$ for linear SVM, and $A$ is a chunk, then we have,*

1. $\lim\limits_{n\to\infty} \frac{\lambda\|\beta_A\|_2^2}{\sum\limits_{x_i \in P} \ell_i(\beta_A)} = 0$;

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2. $\lim\limits_{n \to \infty} \dfrac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in P} \ell_i(\beta_A)} = 0.$

Using this lemma, which we will prove soon, we can prove Theorem 6 for the linear SVM: The definition of coreset allows us to choose any $\beta$, so we can set $\beta = \beta_A$ for a chunk $A$. Then, by Observation 1, Eqn. (2) simplifies to:

$$\frac{|\sum_{x_i \in X} f_i(\beta_A) - \sum_{x_i \in C} u_i f_i(\beta_A)|}{\sum_{x_i \in X} f_i(\beta_A)} \geq \frac{|\sum_{x_i \in X} \ell_i(\beta_A) - \sum_{x_i \in C} u_i \ell_i(\beta_A)| - 2\epsilon\lambda \|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A) + \lambda \|\beta_A\|_2^2}$$

$$= \frac{|1 - \frac{\sum_{x_i \in C} u_i \ell_i(\beta_A)}{\sum_{x_i \in X} \ell_i(\beta_A)}| - \frac{2\epsilon\lambda\|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A)}}{1 + \frac{\lambda\|\beta_A\|_2^2}{\sum_{x_i \in X} \ell_i(\beta_A)}},$$

which tends to 1 as $n \to \infty$ by Lemma 11. This implies that $(C, u)$ is not an $\epsilon$-coreset for the circle instance, which is a contradiction. This completes the proof of Theorem 6 for SVM.

**Proof of Lemma 11**

The remainder of this section is devoted to proving Lemma 11. The proof is very similar to the proof of Lemma 7 and 10.

*Proof.* **of Claim 1 in Lemma 11** We know for all points in $A$, $\ell_i(\beta_A) \geq 1$ this is because all of them have been incorrectly classified. We also know that since $A$ is a chunk, $|A| = \frac{n}{4k}$.

Therefore

$$\sum_{x_i \in A} \ell_i(\beta_A) \geq \frac{n}{4k}$$

We also know $\|\beta_A\|_2 = \sqrt{\frac{n^{1-\gamma}}{k\lambda}}$, so we can conclude

$$\frac{\lambda \|\beta_A\|_2^2}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} \leq \frac{\lambda \|\beta_A\|_2^2}{\sum\limits_{x_i \in A} \ell_i(\beta_A)} \leq \frac{\lambda \|\beta_A\|_2^2}{\frac{n}{4k}} = \frac{\lambda \frac{n^{1-\gamma}}{k\lambda}}{\frac{n}{4k}} = \frac{4}{n^\gamma}$$

The lemma follows by taking the limit of the above inequality. $\qquad\square$

*Proof.* **of Claim 2 in Lemma 11**. Using Observation 8 and the fact that all the points in the coreset are predicted correctly by $\beta_A$ we have:

$$\frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} \leq \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \frac{\|\beta_A\| d_i}{2}\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

Then, by Observation 9 we have

$$\frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} \leq \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \frac{\|\beta_A\|}{2}(\cos\theta - \cos\theta_i)\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

By definition of chunk, we know all the points in $C$ are at least $\frac{n}{4k}$ away from the center of $A$, which means the closest point in $C$ to chunk $A$ is at least $\frac{n}{8k}$ points away, we have $\theta_i \geq \theta + \frac{2\pi}{n}\frac{n}{8k} = \frac{\pi}{2k}$. Therefore,

$$\frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} \leq \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \frac{\|\beta_A\|}{2}(\cos\frac{\pi}{4k} - \cos\frac{\pi}{2k})\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

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Using the Taylor expansion of $\cos(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$, we have,

$$\cos(\frac{\pi}{4k}) - \cos(\frac{\pi}{2k}) \geq \frac{1}{2}((\frac{\pi}{2k})^2 - (\frac{\pi}{4k})^2) - O(\frac{1}{k^4}) = (\frac{3\pi^2}{32k^2}) - O(\frac{1}{k^4})$$

Therefore, we derive,

$$\frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} \leq \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \frac{\|\beta_A\|}{2}((\frac{3\pi^2}{32k^2}) - O(\frac{1}{k^4}))\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

$$= \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \frac{n^{2/5}}{2c\lambda^{2/5}}(\frac{3\pi^2\lambda^{2/5}}{32cn^{2/5-2\gamma}} - O(\frac{\lambda^{4/5}}{n^{4/5-4\gamma}}))\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

$$= \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \alpha n^{2\gamma} + O(\frac{\lambda^{2/5}}{n^{2/5-4\gamma}})\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)}$$

For large enough $n$, we have $\max\left(0, 1 - \alpha n^{2\gamma} + O(\frac{\lambda^{2/5}}{n^{2/5-4\gamma}})\right) = 0$. Therefore, by taking the limit we have:

$$\lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \ell_i(\beta_A)}{\sum\limits_{x_i \in X} \ell_i(\beta_A)} \leq \lim_{n \to \infty} \frac{\sum\limits_{x_i \in C} u_i \max\left(0, 1 - \alpha n^{2\gamma} + O(\frac{\lambda^{2/5}}{n^{2/5-4\gamma}})\right)}{\sum\limits_{x_i \in A} \ell_i(\beta_A)} = 0$$

$\square$

# 6 Experiments

We next experimentally evaluate the practical utility of our uniform sampling scheme for logistic regression. Using several real-world datasets from the UCI machine learning dataset repository [6], we uniformly generate samples of different sizes and train a logistic regression model.

Logistic regression models are trained using mlpack [4]. Given $\lambda = 0.1$, we sweep sample sizes from 50 points up to the dataset size, reporting the mean approximation $H(\beta)$ (according to Eqn. 2) in Figure 1 for a variety of datasets. When training the models, the L-BFGS optimizer is used until convergence [13]. However, although this works for our experiments, note that in general it is not feasible to use L-BFGS like this, specifically when datasets are very large, or when we are in restricted computation access models, as we have considered in this paper. This is because a single L-BFGS step requires computation of the gradient of $f_i(\beta)$ for *every* $i \in [n]$.

In extremely-large-data or streaming settings, a typical strategy for training a logistic regression model is to use mini-batch SGD [22], where the model's parameters are iteratively updated using a gradient computation on a small batch of random points. However, SGD-like optimizers can converge very slowly in practice and have a number of parameters to configure (learning rate, number of epochs, batch size, and so forth). But because our theory allows us to choose a sufficiently small sample, we can use a full-batch optimizer like L-BFGS and this often converges to a much better solution orders of magnitude more quickly.

To demonstrate this, we train a logistic regression model on a sample using L-BFGS and on the full dataset using SGD for 20 epochs. At each step of the optimization, we record the wall-clock time and compute the loss on the full training set (the loss computation time is not included in the wall-clock time). Figure 2 shows the results for three trials of each strategy on two moderately-sized datasets. It is clear from these results that a full-batch gradient descent technique can provide a good approximation of the
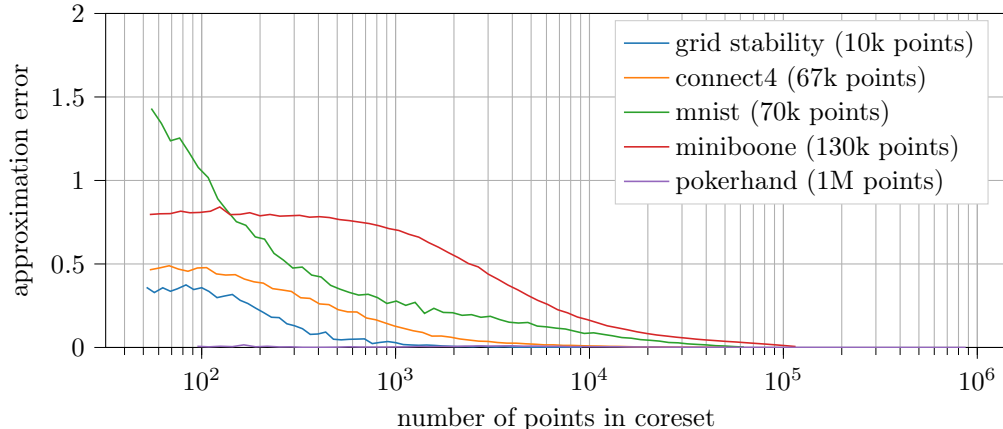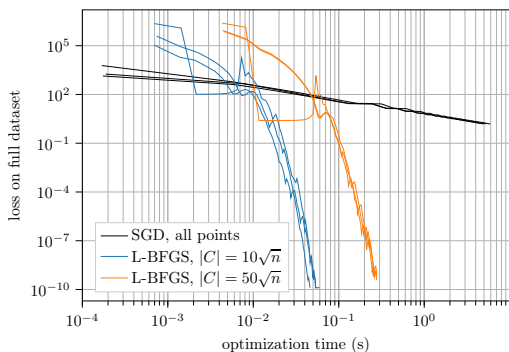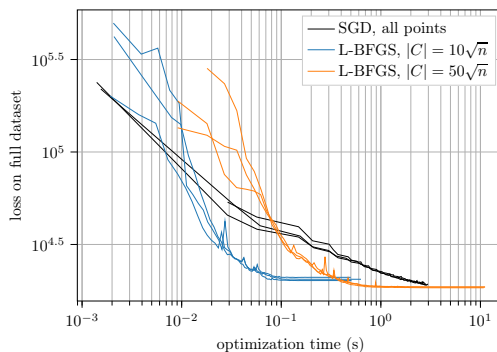
Figure 1: sample approximation error vs. sample size for different datasets.



(a) `covertype` dataset (581k points).

(b) `mnist` dataset (70k points).

Figure 2: Learning curves; log-log axes. Three trials of each strategy are shown. Note the orders-of-magnitude faster convergence for L-BFGS on samples.

full-dataset model with orders-of-magnitude speedup; in fact, L-BFGS is often able to recover a much better model than even 20 epochs of SGD!

To elaborate our experiments we used 5 datasets from the UCI dataset repository [6] of various sizes and dimensionalities. These datasets are collected from synthetic and real-world data sources, and so represent a reasonable collection of diverse datasets. Table 1 gives details on the number of points ($n$) and the number of dimensions ($d$) for each dataset.

In addition, we run three trials of uniform sampling with three different coreset sizes: $\sqrt{n}$, $10\sqrt{n}$, and $20\sqrt{n}$. We plot the relative difference in loss measures (0 means a perfect approximation). Specifically, the approximation given in the table, $H(\beta)$, is given as

$$H(\beta) = \frac{\left| \sum_{i=1}^{n} f_i(\beta) - \sum_{i \in C} u_i f_i(\beta) \right|}{\sum_{i=1}^{n} f_i(\beta)}. \tag{14}$$

We report the mean of $H(\beta)$ over three trials: $\overline{H}(\beta)$.

Overall, we can see that our theory is empirically validated: uniform sampling provides samples that give good empirical approximation, and the use of these samples can result in orders-of-magnitude speedup for learning models. Thus, our theory shows and our experiments justify that uniform sampling to obtain coresets is a compelling and practical approach for restricted access computation models.

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| dataset | $n$ | $d$ | $\|C\| = \sqrt{n}$ | coreset error $\overline{H}(\beta)$ $\|C\| = 10\sqrt{n}$ | $\|C\| = 20\sqrt{n}$ |
|---|---|---|---|---|---|
| connect4 | 67557 | 126 | $0.35 \pm 0.01$ | $0.04 \pm 0.00$ | $0.02 \pm 0.00$ |
| grid_stability | 10000 | 12 | $0.52 \pm 0.00$ | $0.02 \pm 0.00$ | $0.01 \pm 0.00$ |
| miniboone | 130064 | 50 | $0.78 \pm 0.01$ | $0.39 \pm 0.00$ | $0.22 \pm 0.00$ |
| mnist | 70000 | 784 | $0.53 \pm 0.02$ | $0.19 \pm 0.02$ | $0.14 \pm 0.00$ |
| pokerhand | 1000000 | 85 | $1.01 \pm 0.00$ | $0.00 \pm 0.00$ | $0.00 \pm 0.00$ |

Table 1: Dataset information and relative approximation of logistic regression objective with coresets $C$ of different sizes. Three trials are used. Coreset error of 0 indicates a very good approximation.

# Acknowledgements

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